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Some Remarks on Mapping from Semigroup onto Anti-ordered Group

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Abstract

In this article we investigate the subset $H = \{x \in S : (\varphi(x), 1) \in \beta\}$ of semigroup S with apartness, the pre-image of the negative cone of anti-ordered group G by a homomorphism φ from semigroup S onto groups G. This subset is a reflexive completely prime order anti-ideal of S.

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1 Introduction

Setting of this investigation is Constructive Mathematics in sense of the following books [2], [4] and [5]. The investigation is a continuation of author's papers [9]-[12]. In article [12] author describe a construction of anti-ordered group by given anti-ordered semigroup and embedding the last into the anti-ordered group: Let $((S, =, \neq), \cdot, \alpha)$ be a commutative anti-ordered semigroup with apartness such that α is closed for the semigroup operation. Then we can construct an anti-ordered group G that there exists a strongly extensional isotone and reverse isotone mapping from S into G. Here we have intention to analyze opposite situation: Let there exists a homomorphism from semigroup $((S, =, \neq), \cdot)$ onto an anti-ordered group $G = ((G, =, \neq), \cdot, \beta)$. Then, there exists a quasi-antiorder relation $\varphi^{-1}(\beta)$ on S, and an anticongruence

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$q = \varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1}$ on S such that S/q is ordered by anti-order determined by $\varphi^{-1}(\beta)$ and S/q is isomorphic to the group G. In this situation we investigate pre-image of negative cone $H = \{x \in S : ((x), 1) \in \beta\}$ of G.

This paper is motivated by a class of Dubreil-Jacotin semigroups, in the classical Semigroup Theory ([6]): $(S, \cdot, \leq)$ is called Dubreil-Jacotin semigroup if there an isotone semigroup-homomorphism of $(S, \cdot, \leq)$ onto a partially ordered group $(G, \cdot, \beta)$ such that the pre-image of the negative cone of G is a principal order ideal of $(S, \leq)$. This concept was introduced in [8] (see also [1] and [3], Theorem 12.1).

Let $S = ((S, =, \neq), \cdot)$ be a semigroup, $G = ((G, =, \neq), \cdot, 1, \beta)$ an ordered group under anti-order β compatible with the group operation and $\varphi : S \longrightarrow G$ a strongly extensional epimorphism. We give a version of the Bigard theorem ([1], [3]) where in this situation we describe pre-image of negative cone H of group G. This subset is a reflexive completely prime order anti-ideal of S.

2 Preliminaries

In this section, following the standard notions and notations in the Constructive Algebra from our articles [9], [10], [11] and [12], readers let us remember some fundamental notions and facts about (quasi-) anti-ordered sets and semigroups.

Let $X = (X, =, \neq)$ be a set with apartness. For a subset Y of X we say that it is *strongly extensional* if $y \in Y$ and $x \in X$ follows $y \neq x \vee x \in Y$ for any $x, y \in X$. For a relation $\tau \subseteq X \times X$ which is consistent and cotransitive i.e. that satisfies the following conditions:

$$\tau \subseteq \neq, \;\; \tau \subseteq \tau * \tau,$$

where '$*$' is the fulfillment operation between relations, we called *quasi-antiorder relation*. For a subset Y of quasi-antiordered set $X = ((X, =, \neq), \tau)$ we say that it is an *order anti-ideal* of X if the following implication $(x \in Y \wedge y \in X) \Longrightarrow ((x, y) \in \tau \vee y \in Y)$ holds for any $x, y \in X$. If a quasi-antiorder relation $\beta (\subseteq X \times X)$ is linear, i.e. if β satisfies the following conditions $\neq \subseteq \beta^{-1} \cup \beta$, we say that it is an *anti-order relation* on set X and for set X we sat that it is *ordered* under this anti-order β or that it is *anti-ordered* under β. If $X = ((S, =, \neq), \cdot)$ is a semigroup, compatibility of the relation β and the semigroup operation '$\cdot$' means

$$(\forall a, b, x \in S)(((ax, bx) \in \beta \vee (xa, xb) \in \beta) \Longrightarrow (a, b) \in \beta).$$

For relation $q \subseteq X \times X$ we say that it is *coequality* on X if it is consistent, symmetric and cotransitive relation on X. In the case, if $X = ((S, =, \neq), \cdot)$

is a semigroup, compatibility of the relation q and the semigroup operation means

$$(\forall a, b, x \in S)(((ax, bx) \in q \vee (xa, xb) \in q) \Longrightarrow (a, b) \in q).$$

In that case, the relation q we called *anti-congruence* on semigroup S. Further on, for a subset Y of a semigroup S we say that it is a *completely prime subset* of S if the following implication $ab \in Y \Longrightarrow a \in Y \vee b \in Y$ holds for any $a, b \in S$. At least, subset Y of semigroup S is *reflexive subset* of S if $ab \in Y$ implies $ba \in Y$ $(a, b \in S)$.

3 Remarks

Remark A:
Let $S = ((S, =, \neq), \cdot)$ be a semigroup, $G = ((G, =, \neq), \cdot, 1, \beta)$ an anti-ordered group compatible with the group operation and $\varphi : S \longrightarrow G$ a strongly extensional epimorphism. It is known (see [10] and [11]) that $\varphi^{-1}(\beta)$ is a quasi-antiorder relation on S (see: [10], Lemma 2 or [11], Theorem 4, Point 1), $q = Coker\varphi = \{(a, b) \in S \times S : \varphi(a) \neq \varphi(b)\}$ is an anticongruence on S, S/q is ordered by anti-order Θ, defined by $(aq, bq) \in \Theta$ if and only if $(a, b) \in \varphi^{-1}(\beta)$ ([10], Lemma 1), and there exists the strongly extensional mapping $\psi : S/q \longrightarrow G$, given by $\psi(aq) = \varphi(a)$ for any $a \in S$, such that it is an injective and embedding isotone and reverse isotone homomorphism from S/q onto $Im\varphi$ $(\subseteq G)$.

Remarks B:
If $((G, =, \neq), \cdot, 1)$ is a group with compatible anti-order relation β on G and $\varphi : S \longrightarrow G$ is a homomorphism from S into G. Let $H = \{x \in S : (\varphi(x), 1) \in \beta\}$. Then:
(1) H is a *strongly extensional subset* of S. Indeed: Suppose that $a \in H$ and $b \in S$, i.e. suppose $(\varphi(a), 1) \in \beta \wedge b \in S$. Then, $(\varphi(a), \varphi(b)) \in \beta \vee (\varphi(b), 1) \in \beta$. Thus, by consistency of β and strongly extensionality of φ, we have $a \neq b \vee b \in H$.

(2) If $\beta \cap \beta^{-1} = \emptyset$, H is a *subsemigroup* of S. In fact, for $a \in H$, we have $(\varphi(a), 1) \in \beta \wedge (\varphi(b), 1) \in \beta$. Thus, $((\varphi(a), \varphi(ab)) \in \beta \vee (\varphi(ab), 1) \in \beta) \wedge (\varphi(b), 1) \in \beta$ and $((1, \varphi(b)) \in \beta \wedge (\varphi(b), 1) \in \beta) \vee (\varphi(ab), 1) \in \beta$. Therefore, we have $(\varphi(ab), 1) \in \beta$ i.e. $ab \in H$.

(3) H is *order anti-ideal* of S. Indeed: For $a \in H$ and $b \in S$, we have $(\varphi(a), 1) \in \beta b \in S$. Thus, $(\varphi(a), \varphi(b)) \in \beta$ or $(\varphi(b), 1) \in \beta$ and $(a, b) \in \varphi^{-1}(\beta) \vee (\varphi(b), 1) \in \beta$. This means $(a, b) \in \varphi^{-1}(\beta) \vee b \in H$.

(4) H is *completely prime subset* of S because for $ab \in H$, i.e. for $(\varphi(ab), 1) \in \beta$, we have $(\varphi(a)\varphi(b), \varphi(b)1) \in \beta \vee (\varphi(b)1, 1) \in \beta$. Thus, $(\varphi(a), 1) \in \beta \vee$

$(\varphi(b), 1) \in \beta$ which means $a \in H \vee b \in H$.

(5) H is *reflexive subset* of S. Indeed, for $ab \in H$, we have $(\varphi(ab), 1) \in \beta$. This is equivalent with $(\varphi(a)\varphi(b), \varphi(b)^{-1}\varphi(b)) \in \beta$. Thus, $(\varphi(a), \varphi(b)^{-1}) \in \beta$. Further on, we have $(\varphi(b)^{-1}\varphi(b)\varphi(a), \varphi(b)^{-1}) \in \beta$. The least we conclude $(\varphi(ba), 1) \in \beta$, which means $ba \in H$.

Finally, according to remarks mentioned above, we can conclude that the subset H is a strongly extensional, reflexive, completely prime, order anti-ideal of S. Especially, if the anti-order β on G satisfies the following condition $\beta \cap \beta^{-1} = \emptyset$, then the set H is a subsemigroup of S.

Remark C:
Let x be an arbitrary element of S such that $\varphi(x) \neq 1$. Then there exists an element y of S such that $\varphi(xy) \neq 1$ because φ is surjective. Thus, $(\varphi(xy), 1) \in \beta$ or $(1, \varphi(xy)) \in \beta$. From the first case, we have $y \in [H : x] = \{u \in S : yu \in H\}$. In the second case, from $(1, \varphi(xy)) \in \beta$ we have $(\varphi(x)^{-1}\varphi(y)^{-1}, 1) \in \beta$ and $(\varphi(x)^{-1}\varphi(y)^{-1}\varphi(x)^{-1}\varphi(x), 1) \in \beta$. Finally, again φ is onto, there exists an element t of S such that $\varphi(t) = \varphi(x)^{-1}\varphi(y)^{-1}\varphi(x)^{-1}$ and we have $t \in [H : x]$. Therefore, for any $x \in S$ such that $\varphi(x) \neq 1$, holds $[H : x] \neq \emptyset$. Since φ is surjective, then there exists an element $t \in S$ such that $\varphi(t) = 1$. Thus, we conclude $(\varphi(t), 1) \bowtie \beta \cup \beta^{-1}$.

Example: Let B be a band, (G, β) an anti-ordered group and let $S = B \times G$ be their direct product with the internal operation '$\circ$' defined by $(e, a) \circ (f, b) = (ef, ab)$. Then, the set $((S, =, \neq), \circ)$ is a semigroup. The natural anti-order γ on S is given by $((e, a), (f, b)) \in \gamma$ if $(e, f) \in \alpha$ or $(a, b) \in \beta$ where α is a natural anti-order in band ([7], Lemma 1). Notice that, γ is not compatible with multiplication operation in S, in general. The projection $\varphi : S \longrightarrow G$, defined by $\varphi((e, a)) = a$, is a reverse isotone epimorphism because the implication $(a, b) \in \beta \Longrightarrow ((e, a), (f, b)) \in \gamma$ holds for any $e, f \in B$. Further on, the relation

$$\varphi^{-1}(\beta) = \{((e, a), (f, b)) : e \in B \wedge f \in B \wedge (a, b) \in \beta\}$$

is a quasi-antiorder relation on S compatible with the semigroup operation. So, the relation

$$\Theta = \{((e, a)q, (f, b)q) \in S/q \times S/q : e \in B \wedge f \in B \wedge (a, b) \in \beta\}$$

is induced anti-order on factor-semigroup S/q, where

$$q = Coker = \{((e, a), (f, b)) : e \in B \wedge f \in B \wedge a \neq b\}.$$

The mapping $\psi : S/q \longrightarrow G$ is isotone and reverse isotone isomorphism and the set $H = \{(\varphi((e, a)), 1) \in B \times G : e \in B \wedge (a, 1) \in \beta\} = \{(e, a) : (a, 1) \in \beta\}$ is an negative cone of S.

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